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On a class of univalent functions

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ABSTRACT

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{N}$ the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left| -z^3 \left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le 1, \quad z \in \mathbb{D}.$$

We show that functions in $\mathcal{N}$ are univalent in $\mathbb{D}$ but not necessarily starlike. Also, we present the characterization formula, necessary and sufficient coefficient conditions for functions to be in the class $\mathcal{N}$.

1. Introduction and main results

Let $\mathcal{H}$ be the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in $\mathcal{H}$. Let $\mathcal{S}$ denote the class of functions f in $\mathcal{A}$ such that f is univalent in $\mathbb{D}$. We consider

$$\mathcal{U} = \left\{f \in \mathcal{A} : \left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le 1,\ z \in \mathbb{D}\right\}$$

$$\mathcal{P} = \left\{f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)''\right| \le 2,\ z \in \mathbb{D}\right\}, \text{ and}$$

$$\mathcal{M} = \left\{f \in \mathcal{A} : |M_f(z)| \le 1,\ z \in \mathbb{D}\right\},$$

where

$$M_f(z) = z^2\left(\frac{z}{f(z)}\right)'' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1.$$

Recently, the authors [1] have studied the class $\mathcal{M}$, and obtained the strict inclusion

$$\mathcal{M} \subsetneq \mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}.$$

Many properties of the classes $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{M}$ and their generalizations have been studied extensively in [2–5,1]. Also, it is well-known that (see [6]) if we set $\mathcal{S}_{\mathbb{Z}} = \{f \in \mathcal{S} : a_n \in \mathbb{Z}\}$, then

$$\mathcal{S}_{\mathbb{Z}} = \left\{z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2}\right\}.$$

Further, it has been verified that $\mathcal{S}_{\mathbb{Z}} \subsetneq \mathcal{U} \cap \mathcal{P} \cap \mathcal{M}$ (see [1, Theorem1]).

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In this article, we consider the class $\mathcal{N}$ of functions $f \in \mathcal{A}$ which satisfy the condition $|N_f(z)| \le 1$ for $z \in \mathbb{D}$, where

$$N_f(z) = -z^3 \left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1. \tag{1}$$

We show that the class $\mathcal{N}$ possesses many interesting properties.

First, we observe that it is a simple exercise to see that $\mathcal{S}_{\mathbb{Z}} \subsetneq \mathcal{N}$ and so, we have the interesting strict inclusion $\mathcal{S}_{\mathbb{Z}} \subsetneq \mathcal{U} \cap \mathcal{P} \cap \mathcal{M} \cap \mathcal{N}$. It is worth remembering that the Koebe function belongs to the class $\mathcal{N}$ and therefore, is of our interest in this paper.

Now, we state our main results and the proofs of these will be given in Section 3.

Theorem 1 (*Inclusion Property*). *We have the strict inclusion $\mathcal{N} \subsetneq \mathcal{M} \cap \mathcal{U} = \mathcal{M}$.*

Example 1. Consider the function f defined by

$$\frac{z}{f(z)} = 1 + \frac{1}{2}z + \frac{\lambda}{2}z^3,$$

where $0 < \lambda \le 1$. Then we see that $z/f(z) \neq 0$ in $\mathbb{D}$. Further

$$f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 = -\lambda z^3, \qquad M_f(z) = 2\lambda z^3, \quad \text{and} \quad N_f(z) = -4\lambda z^3.$$

Thus, if $1/2 < \lambda \le 1$, then we see that $f \in \mathcal{U}$ whereas $f \notin \mathcal{M}$ and $f \notin \mathcal{N}$. Thus, there exists a function $f \in \mathcal{U}$ such that f is neither in $\mathcal{N}$ nor in $\mathcal{M}$.

Theorem 2 (*Sufficiency Coefficient Condition*). *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in $\mathbb{D}$ and that it satisfies the coefficient condition*

$$\sum_{n=2}^{\infty} (n-1)^3 |b_n| \le 1. \tag{2}$$

Then the function f defined by $f(z) = z/\phi(z)$ is in $\mathcal{N}$.

For example, according to (2), each function in $\mathcal{S}_{\mathbb{Z}}$ belongs to $\mathcal{N}$.

Let $\mathcal{S}^*$ denote the class of univalent functions in $f \in \mathcal{S}$ such that the range $f(\mathbb{D})$ is a starlike domain (with respect to the origin). Analytically, $f \in \mathcal{S}^*$ if and only if $\mathrm{Re}(zf'(z)/f(z)) > 0$ in $\mathbb{D}$. As $\mathcal{N} \subsetneq \mathcal{M}$, it is natural to ask whether the class $\mathcal{N}$ is included in $\mathcal{S}^*$. Our computation leads to the following conjecture, although we are not able to prove it for the moment.

Conjecture 1. *Neither the class $\mathcal{M}$ nor the class $\mathcal{N}$ is included in $\mathcal{S}^*$.*

If f and g are analytic functions on $\mathbb{D}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the convolution (Hadamard product) of f and g, denoted by $f * g$, is an analytic function on $\mathbb{D}$ given by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Although $\mathcal{U}$ is neither included in $\mathcal{N}$ nor in $\mathcal{M}$, in the following result, we show that the classes $\mathcal{U}$ and $\mathcal{M}$ can be used to extract functions to belong to $\mathcal{N}$.

Theorem 3 (*Multiplier Theorem*). *Let $f \in \mathcal{U}$ and $g \in \mathcal{M}$ have the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \textit{and} \quad \frac{z}{g(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{3}$$

and such that $\frac{z}{f(z)} * \frac{z}{g(z)} \neq 0$ on $\mathbb{D}$. Then the function H defined by

$$H(z) = \frac{z}{(z/f(z)) * (z/g(z))}$$

is in the class $\mathcal{N}$. More generally, if $f \in \mathcal{U}$ and $g \in \mathcal{P}$, then $H \in \mathcal{N}$. In particular, if $f, g \in \mathcal{M}$ then $H \in \mathcal{N}$.

Corollary 1 (*Necessary Coefficient Condition*). *Let $f \in \mathcal{N}$ and have the form (3). Then we have*

$$\sum_{n=2}^{\infty} (n-1)^6 |b_n|^2 \le 1.$$

Proof. As in the proofs of Theorems 2 and 3, we see that

$$N_f(z) = -\sum_{n=2}^{\infty}(n-1)^3 b_n z^n,$$

where $N_f(z)$ is defined by (1), and therefore, we easily have the desired necessary condition. $\square$

Theorem 4 (*Characterization Theorem*)**.** *Every $f \in \mathcal{N}$ has the representation*

$$\frac{z}{f(z)} = 1 - \frac{f''(0)}{2}z + \int_0^1 \frac{(\log(1/t))^2}{t^2} w(tz)dt,$$

for some $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = w'(0) = 0$.

2. Preliminary lemmas

Let $\mathcal{P}_n$ denote the class of functions p in $\mathcal{H}$ such that $p^{(k)}(0) = 0$ for $k = 0, 1, 2, \dots, n$, where $p^{(0)}(0) = p(0)$. With $w^{(0)}(z) = w(z)$, we set

$$\mathcal{B}_n = \{w \in \mathcal{H} : |w(z)| \le 1,\ w^{(k)}(0) = 0 \text{ for } k = 0, 1, \dots, n\}.$$

Lemma 1. *Let $p \in \mathcal{P}_1$. If p satisfies the condition*

$$\left|p(z) + (\gamma - 2\sqrt{\gamma})zp'(z) + \gamma z^2 p''(z)\right| \le 1, \quad z \in \mathbb{D}, \tag{4}$$

for some $\gamma > 1/4$, then we have the following:

(i) $|p(z)| \le \frac{|z|^2}{(2\sqrt{\gamma}-1)^2}$, $z \in \mathbb{D}$,

(ii) $|-zp'(z) + p(z)| \le \left(\frac{1}{\sqrt{\gamma}(2\sqrt{\gamma}-1)} + \left|1 - \frac{1}{\sqrt{\gamma}}\right| \frac{1}{(2\sqrt{\gamma}-1)^2}\right)|z|^2$, $z \in \mathbb{D}$.

In particular,

$$\left|p(z) - zp'(z) + z^2 p''(z)\right| \le 1 \Longrightarrow |p(z)| \le |z|^2 \quad \text{and} \quad |-zp'(z) + p(z)| \le |z|^2. \tag{5}$$

Proof. First, we rewrite (4) as

$$p(z) + (\gamma - 2\sqrt{\gamma})zp'(z) + \gamma z^2 p''(z) = w(z), \tag{6}$$

where $w \in \mathcal{B}_1$. Now, we let

$$p(z) = \sum_{k=2}^{\infty} p_k z^k \quad \text{and} \quad w(z) = \sum_{k=2}^{\infty} w_k z^k.$$

A comparison of the coefficients of z^k on both sides in (6) gives that

$$p_k = \frac{w_k}{(k\sqrt{\gamma} - 1)^2} \quad \text{for } k \ge 2. \tag{7}$$

Using this, we see that

$$p(z) = \frac{1}{\gamma}\sum_{k=2}^{\infty} \frac{w_k}{(k - (1/\sqrt{\gamma}))^2} z^k.$$

Now, we recall that (see for example [7])

$$\sum_{k=1}^{\infty} \frac{1}{(k+a)^2} z^k = z\int_0^1 \frac{t^a \log(1/t)}{1 - tz}\, dt \quad \text{for } a > -1$$

from which we easily obtain that

$$\sum_{k=2}^{\infty} \frac{1}{(k+a)^2} z^k = z^2 \int_0^1 \frac{t^{a+1} \log(1/t)}{1 - tz}\, dt \quad \text{for } a > -2.$$

Using this observation, it follows that for $\gamma > 1/4$

$$\begin{aligned} p(z) &= \frac{1}{\gamma} w(z) * \sum_{k=2}^{\infty} \frac{1}{(k-(1/\sqrt{\gamma}))^2} z^k \\ &= \frac{1}{\gamma} w(z) * z^2 \int_0^1 \frac{t^{1-(1/\sqrt{\gamma})} \log(1/t)}{1-tz}\, dt \\ &= \frac{1}{\gamma} \int_0^1 t^{-1-(1/\sqrt{\gamma})} \log(1/t) w(tz)\, dt. \end{aligned}$$

As $w \in \mathcal{B}_1$, Schwarz' lemma gives that $|w(z)| \le |z|^2$ in $\mathbb{D}$, and therefore, we conclude that

$$\begin{aligned} |p(z)| &\le \frac{1}{\gamma} |z|^2 \int_0^1 t^{1-(1/\sqrt{\gamma})} \log(1/t)\, dt \\ &= \frac{1}{\gamma} |z|^2 \frac{1}{(2-(1/\sqrt{\gamma}))^2} = \frac{|z|^2}{(2\sqrt{\gamma}-1)^2} \end{aligned}$$

and the conclusion (i) follows.

For the proof of (ii), by (7), we can easily deduce that

$$\begin{aligned} -zp'(z) + p(z) &= -\frac{1}{\gamma} \sum_{k=2}^{\infty} \frac{(k-1)w_k}{(k-(1/\sqrt{\gamma}))^2} z^k \\ &= -\frac{1}{\gamma} \left(\sum_{k=2}^{\infty} \frac{w_k}{k-(1/\sqrt{\gamma})} z^k + \left(\frac{1}{\sqrt{\gamma}} - 1 \right) \sum_{k=2}^{\infty} \frac{w_k}{(k-(1/\sqrt{\gamma}))^2} z^k \right) \\ &= -\frac{1}{\gamma} \int_0^1 t^{-1-(1/\sqrt{\gamma})} w(tz)\, dt - \left(\frac{1}{\sqrt{\gamma}} - 1 \right) \frac{1}{\gamma} \int_0^1 t^{-1-(1/\sqrt{\gamma})} \log(1/t) w(tz)\, dt. \end{aligned}$$

Again as $|w(z)| \le |z|^2$ in $\mathbb{D}$, we obtain that

$$|-zp'(z) + p(z)| \le \frac{|z|^2}{\gamma(2-(1/\sqrt{\gamma}))} + \left| 1 - \frac{1}{\sqrt{\gamma}} \right| \frac{|z|^2}{\gamma(2-(1/\sqrt{\gamma}))^2}$$

and the conclusion (ii) follows. □

3. Proofs

Proof of Theorem 1. Let $f \in \mathcal{N}$ and set

$$p(z) = \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 = -z \left(\frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1.$$

Then p is analytic in $\mathbb{D}$, $p(0) = p'(0) = 0$,

$$p(z) - zp'(z) + z^2 p''(z) = N_f(z) \quad \text{and} \quad -zp'(z) + p(z) = M_f(z),$$

where N_f is defined by (1) and

$$M_f(z) = z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1.$$

Now, as $f \in \mathcal{N}$, we obtain that

$$|p(z) - zp'(z) + z^2 p''(z)| \le 1, \quad z \in \mathbb{D}.$$

If we apply Lemma 1 with $\gamma = 1$, namely, the implication (5), it follows that

$$|p(z)| \le |z|^2 \quad \text{and} \quad |-zp'(z) + p(z)| \le |z|^2, \quad z \in \mathbb{D}$$

and therefore, $f \in \mathcal{U}$ and $f \in \mathcal{M}$. It has been shown in [1, Theorem 1] that $\mathcal{M} \subsetneq \mathcal{U}$ and so, $\mathcal{M} \cap \mathcal{U} = \mathcal{M}$. □

Proof of Theorem 2. Let f be given by $f(z) = z/\phi(z)$, where $\phi(z) \neq 0$ in $\mathbb{D}$ and $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Since

$$-z \left(\frac{z}{f(z)} \right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)} \right)^2 f'(z),$$

we have

$$N_f(z) = -z^3\left(\frac{z}{f(z)}\right)''' - \left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 = -\sum_{n=2}^{\infty}(n-1)^3 b_n z^n.$$

Thus, using the coefficient condition (2), we deduce that

$$\left|N_f(z)\right| \le \sum_{n=2}^{\infty}(n-1)^3|b_n|\,|z|^n \le \sum_{n=2}^{\infty}(n-1)^3|b_n| \le 1$$

and therefore, $f \in \mathcal{N}$. □

Proof of Theorem 3. Suppose that $f \in \mathcal{U}$ and $g \in \mathcal{M}$. By hypotheses, $\frac{z}{H(z)} \neq 0$ for $z \in \mathbb{D}$. Using the power series representation of f, we obtain that

$$\left|-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1\right| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| \le 1.$$

Therefore, as in [1], we let $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \le \theta \le 2\pi$ so that the last inequality gives

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right|^2 d\theta \le 1.$$

Allowing $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \le 1. \tag{8}$$

Similarly, as $g \in \mathcal{M}$, the power series representation of g gives

$$M_g(z) = \sum_{n=2}^{\infty}(n-1)^2 c_n z^n$$

and so, as above, one has

$$\sum_{n=2}^{\infty}(n-1)^4|c_n|^2 \le 1. \tag{9}$$

Now, since

$$\frac{z}{f(z)} * \frac{z}{g(z)} = 1 + b_1c_1z + b_2c_2z^2 + \cdots.$$

Eqs. (8) and (9) give

$$\sum_{n=2}^{\infty}(n-1)^3|b_n|\,|c_n| \le \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)^4|c_n|^2\right)^{1/2} \le 1.$$

Finally, by (2), we conclude that $H \in \mathcal{N}$. □

Proof of Theorem 4. Let $f \in \mathcal{N}$ with $a_2 = f''(0)/2$. If we let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, then $z/f(z)$ takes the form

$$\frac{z}{f(z)} = 1 - a_2z - (a_3 - a_2^2)z^2 - (a_4 - 2a_2a_3 + a_2^3)z^3 + \cdots.$$

Now, we find that

$$N_f(z) = -(a_3 - a_2^2)z^2 - 4(a_4 - 2a_2a_3 + a_2^3)z^3 + \cdots = w(z)$$

where $w \in \mathcal{B}_1$. Also, we see that

$$N_f(z) = p(z) - zp'(z) + z^2p''(z) = w(z) \tag{10}$$

with

$$p(z) = -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)}.$$

We may now set $w(z) = \sum_{k=2}^{\infty} w_k z^k$. From the proof of Lemma 1, it follows from (10) that

$$p(z) = \sum_{k=2}^{\infty} \frac{w_k}{(k-1)^2} z^k = \int_0^1 t^{-2} \log(1/t) w(tz)\, dt.$$

Then the last two relations give (for example using the comparison of the coefficients)

$$\begin{aligned} \frac{z}{f(z)} &= 1 - a_2 z + \sum_{k=2}^{\infty} \frac{w_k}{(k-1)^3} z^k \\ &= 1 - a_2 z + w(z) * \sum_{k=1}^{\infty} \frac{z^{k+1}}{k^3} \\ &= 1 - a_2 z + w(z) * z^2 \int_0^1 \frac{(\log(1/t))^2}{1-tz}\, dt \quad \text{(see [7])} \\ &= 1 - a_2 z + \int_0^1 \frac{(\log(1/t))^2}{t^2} w(tz)\, dt \end{aligned}$$

and the desired representation follows. □

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